A Generalized Relational Model for Demographic Analysis

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Describing and analysing age patterns of vital events is one of the fundamental concerns of demography. Since the time the documentation of vital events began, the shape of mortality curve by age has puzzled the inquisitive mind. Its shape tended to vary with space and time. When attempts to capture the age variation in mortality by a single mathematical formula with only a few parameters proved elusive, demographers began constructing and tabulating 'model' life tables from empirical data. Such tabulations of model life tables have found extensive use in population projections and in making demographic estimation from incomplete or defective data.

Continuing in the tradition of describing the age curve of mortality though mathematical functions, Brass (1971) discovered that it is possible to achieve a linear representation of the differences in the survivorship probabilities \( l_x \) in any two life tables with the help of only two parameters. The Brass relational logit system has the form:

\[
\logit(l_x) = \alpha + \beta \logit(l_{sx})
\]

where \( \logit(l_x) = \frac{1}{2} \ln \left( \frac{1-l_x}{l_x} \right) \),

and \( l_{sx} \) represents the survivorship probabilities of the 'standard' life table, i.e., the life table with which other life tables are compared. The model parameter \( \alpha \) is an indicator of the level of mortality that affects all ages about equally, while \( \beta \) modifies this effect by age. Values of \( \beta > 1 \) imply, relative to the standard, lower mortality at younger ages but higher mortality at older ages, while values of \( \beta < 1 \) imply the vice versa.

However, repeated applications of the model have shown that the logit transformation doesn’t fully capture the complexity of variations in levels and age patterns of mortality found in human populations. In particular, it has been observed that deviations from linearity become large when the level of mortality of a population is far from the standard. Attempts to modify the logit model by including additional parameters have proved to be of little practical significance because the data required to estimate the additional parameters are not usually available (Zaba 1979; Ewbank et al. 1983). But recently, Murray and his colleagues at the World Health Organization have proposed a modification that largely remedies the problems of earlier attempts (Murray et al. 2003). Instead of complicating the model by including additional parameters, they suggest estimating the model deviations from linearity from the available information on the levels of child and adult mortality for the population and an empirically derived standard pattern of deviations.

But the success of the WHO approach is achieved with a cost. Because the standard pattern of deviations has been defined in terms of a standard survivorship function (the WHO standard hereafter) it sacrifices the option of using different standard survivorship functions in
the Brass model. While the use of a single standard survivorship function is appealing from the
point of view of grand generalization, and eliminating arbitrariness in the choice of the standard
life table, this flexibility is useful to have in individual country applications. For, as we shall
show, an appropriately chosen standard survivorship function can improve the model fit even
more than what the WHO approach could achieve with the same data.

In this paper an attempt is made to analyse the reasons for deviations from linearity that
the Brass logit model exhibits as mortality levels depart from the standard, and to propose a
parametric modification to the model that would minimize these deviations. The suggested
modification retains the flexibility of using different standard survivorship functions while the
additional parameter that it incorporates can be appropriately fixed, if available data doesn’t
permit its estimation. The proposal is based on a generalized family of growth curves of which
the logit and the Gompertz are special or limiting cases. While this generalized curve has
applicability in other areas of demography such as in the analysing the age patterns fertility and
marriage, the paper is restricted to elucidate its relevance to mortality analysis.

Accounting for Departure from Linearity

Brass’s proposal of the logit model was based on the idea that when the probability
distribution of deaths by age is suitably transformed, a single peaked, symmetrical distribution
would result (Brass 1971). Brass identified logit as the required transformation. If we denote
logit \( l_x \) as \( y \), and drop the scalar multiplier \( \frac{1}{2} \), eq. (1) can be written as

\[
\ln \left[ \frac{1-l_x}{l_x} \right] = a + b y
\]

\[
l_y = \frac{1}{1 + e^{a+by}}
\]

By differentiation we have

\[
\frac{\partial l_y}{\partial y} = -b l_y (1-l_y)
\]

Eq. (2) is essentially the differential equation of the logistic curve, which has a symmetrical
shape, and a point of inflection (mode) at \( l_y = 0.5 \). In discrete notations, eq. (2) can be rewritten as

\[
\logit(l_{x+n}) - \logit(l_x) = -b l_{x+n/2} (1-l_{x+n/2})
\]

where \( n \) is the life table deaths in the age interval \( x \) to \( x+n \). Thus the Brass model assumes that
when the density distribution of deaths of a life table is expressed in the intervals of the chosen
logit standard (i.e., in \( y \) instead of \( x \)) and plotted against the same life table’s survivorship
function, it would have a symmetrical shape.

But does this happen in practice? To test this, female West model life tables with life
expectancy at birth (\( e_0 \)) of 45, 55, 65 and 75 were chosen. The frequency distribution of deaths of
each life table was 'standardized' by dividing $nd_x$ by the difference in the logit of $l_x$ and $l_{x+n}$ of the life table with $e_0 = 45$. Figure 1 shows the plot of the deaths in the standardized age scale against the survivors at the midpoint of the age interval (taken to be the geometric mean of $l_x$ and $l_{x+n}$) in the respective life table.

Since the death distribution of the life table of $e_0 = 45$ was standardized using its own survivorship function, it does have a symmetrical shape and a point of inflection at $l_x = 0.5$. But standardized death distributions at higher life expectancies do not show symmetrical shapes. As the level of life expectancy increases, the mode of the death distribution shifts towards higher $l_x$, that is, towards younger ages.

What happens to the shape of the death distribution if, instead of $l_x$ of $e_0 = 45$, that of $e_0 = 75$ is used as the standard? The graph at the lower panel of Figure 1 shows its impact. Although the distributions of deaths of $e_0 = 75$ and $e_0 = 65$ have inverted U-shapes, those of 45 and 55 are highly skewed, with the mode again shifting towards younger ages. Clearly, although the standardization using the logit of survivorship function does stretch the age scale in the required direction, its performance depends on the chosen standard.

The tendency of the standardized distribution of deaths becoming skewed as the level of mortality deviates from the standard life table is not just limited to model life table systems. It is also observed in real world. Figure 2 shows the standardized destiny distribution of deaths for US males for selected years using the survivorship functions of 1900 and 1993 as standards. Again, the change in the shape of the death distribution as mortality levels change is similar to that observed in the West model tables. In fact, the shift is even more pronounced.

The Gompertz Model

The foregoing analysis indicates that the limitation of the logit transformation arise from the fact that it implies a symmetric distribution of deaths in the transformed age scale. Perhaps other S-shaped curves that have skewed density distributions could do this job better. The Gompertz is one such curve. But the Gompertz transformation of $l_x$ would be skewed towards the wrong direction, i.e., the mode would be shifted to the lower end of survivorship function. However, the Gompertz transformation of $q_x$ (i.e., $1 - l_x$) would have the point of inflection in the right direction. The density distribution of this Gompertz curve is given by

$$\frac{\partial q_y}{\partial y} = b q_y (-\ln q_y)$$

where $y$ indexes age ($x$) in the transformed scale. The point of inflection of this curve occurs at $q_y = 1/e$, or when $l_y$ is approximately equal to 0.63.

By integrating eq. (3) we have,

$$\ln(-\ln(q_y)) = a - by$$

for $y > 0$

where $a = \ln(-\ln(q_0))$.

We may therefore write the relational Gompertz model for $q_x$ as
\[
\ln(-\ln(q_x)) = \alpha + \beta \ln(-\ln(q_{xx})).
\] (4)

Brass himself had suggested this transformation for modelling age patterns of fertility. But for unclear reasons, he has not chosen the Gompertz transformation for mortality analysis. At this point, we should also note that it is theoretically more appropriate to write the corresponding logit model in \( q_x \) rather than in \( l_x \):

\[
\text{logit}(q_x) = \ln \left( \frac{1 - q_x}{q_x} \right) = \alpha + \beta \ln \left( \frac{1 - q_{xx}}{q_{xx}} \right)
\] (5)

because \( q_x \) represents the cumulative density distribution of deaths in the life table. As the density distribution of the logistic curve is symmetric, it does not matter whether the logit transformation is performed on the \( l_x \) or \( q_x \). But for asymmetric distributions this distinction is critical.

**Generalized Logit Model**

Both logistic and Gompertz curves have fixed inflection points. Consequently, when the parameter \( \beta \) of the relational model is changed, the \( l_x \) curves intersect at a fixed point in the survivorship function (0.5 in the logit model and 0.63 in the Gompertz model). Since the mode of the density distribution of deaths could be shifting as mortality levels change, one may get better fit with a curve having variable inflection point. Although not widely known, a generalized logistic curve that accommodates the Gompertz curve as a limiting case has been proposed several decades ago (Von Bertalanffy 1957). In the notations used this paper the differential equation of this curve can be written as

\[
\frac{\partial q_y}{\partial y} = b \ q_y^{1-v} (1 - q_y^v)
\] (6)

The integration of eq. (6) yields

\[
q_y = \left[ 1 - (1 - q_0^v) e^{-by} \right]^{1/v}
\] (7)

where \( q_0 \) is \( q_y \) when \( y = 0 \).

It may be noted that when \( v=1 \), the cumulative density function reduces to the 'confined' exponential:

\[
q_y = 1 - (1 - q_0) e^{-by}
\]

and when \( v=-1 \) the equation defines the logistic curve:

\[
q_y = \left[ 1 + (q_0^{-1} - 1) e^{-by} \right]^{-1}
\]
When $v=0$, the function is not defined. But it can be shown that as $v \to 0$, the model approaches the Gompertz function (for proof, see Mahajan and Peterson 1985).

It can also be easily shown that the point of inflection of this curve is at

$$q_y = (1 - v)^{\frac{1}{v}}, \quad \text{for } v < 1.$$  

Thus the inflection point can lie anywhere between 0 and 1.

The relational model under this curve can be written as

$$\ln(q_x^v - 1) = \alpha + \beta \ln(q_x^v - 1) \quad \text{for } v < 0,$$

and

$$\ln(1 - q_x^v) = \alpha + \beta \ln(1 - q_x^v) \quad \text{for } 0 < v < 1.$$  

To gain clearer understanding of the role played by the new parameter $v$, Figure 3 shows the $l_x$ curves generated by the relational system under different values of $\alpha$, $\beta$ and $v$. First, it should be noted that when both $\alpha$ and $\beta$ are held constant, changing $v$ will not have any effect on the survivorship curve. The graph at the top panel of Figure 3 shows the impact of changing both $\beta$ and $v$, with $\alpha$ held constant. The resulting $l_x$ curves intersect the standard at different points depending on the value of $v$, when $v < 0$. But survivorship functions generated by the positive values of $v$ will not intersect the standard.

The relation that defines the point of intersection of the $l_x$ curve is given by

$$\ln(q_{xx}^v - 1) = 0, \text{ or when}$$

$$q_{xx} = 2^{\frac{1}{v}}, \quad v < 0.$$  

Because of this property, under this system, it is possible to generate survivorship curves that intersect at any given $l_x$ value by appropriately setting the value of $v$. The value of $v$ to be set is given by the equation

$$v = \frac{\ln 2}{\ln(1 - l_x)}.$$  

The middle panel of Figure 3 shows the $l_x$ curves generated such that they intersect the standard $l_x$ curve at age 5. That is, all the curves have the same $\alpha$, $v$ and $l_5$, but different values of $\beta$.

Finally, the bottom panel of Figure 3 shows the effect of changing $\alpha$ with different values of $v$. As the figures shows, this results in non-intersecting $l_x$ curves, but as $v$ increases, the effect of changing $\alpha$ is enhanced.

By how much does the new models reduce the departure from linearity observed under
the logit model? Figure 4 shows the impact on the survivorship functions of US males from 1900 to 1993, by taking \( l_x \) of 1900 as the standard. The graph at the top panel shows the transformation using \( v = -1 \), which reduces the model to the logit of \( q_x \). The departure from linearity that Murray et al. had observed is clearly reflected in the graph. But as the values of \( v \) increases, the lines become more linear, though the bend at the youngest ages does not completely disappear. The last panel in the figure shows the effect of the Gompertz transformation. It is clear that it performs better than the logit model.

But the graphs could be deceiving. A better check is to perform linear regressions and compare the fit under different models. Accordingly, Table 1 presents the results of such regressions for the life table of 1993 using that of the 1900 and 1960 as standards. As the values of \( R^2 \) and \( F \)-statistic show, the fit improves significantly when the Gompertz model is used and even more so when \( v \) is set to equal 0.5 under the generalized logit model. It is also observed that all the models perform better when the standard is switched to 1960 from 1900. Thus, the flexibility to change the standard is a useful option to have.

**Model Fitting when Data are Incomplete**

A principal use of these models is for generating a complete life table when only partial information is available. The most typical situation is one of having information on under-5 mortality (\( q_5 \)) and an estimate of adult mortality, such as \( 45q_{15} \). In such circumstances, it is possible only to estimate two model parameters. Thus under the generalized logit model one of the parameters must be held constant at some level. The obvious choice is to hold \( v \) fixed, but not necessarily at -1, which would reduce the equation to the Brass model. The method recommended here takes advantage of the model's ability to fix \( v \) in such a way that the generated \( l_x \) curves would intersect at a chosen age when \( \beta \) is changed. The steps of this iterative method is as follows:

1. Choose an appropriate standard life table.
2. For generating \( l_x \) functions having the same \( l_5 \), fix \( v \) using the relation
   \[
   v = \frac{\ln 2}{\ln(1 - l_5)}.
   \]
3. Assume a starting value for \( \beta \) (\( \beta = 1 \) is suggested).
4. Estimate \( \alpha \) as
   \[
   \alpha = \ln(q_5^v - 1) - \beta \ln(q_5^v - 1) \quad \text{if } v < 0,
   \]
   \[
   \alpha = \ln(1 - q_5^v) - \beta \ln(1 - q_5^v) \quad \text{if } v > 0.
   \]
5. Compute the survivorship function as
   \[
   l_x = 1 - \left[ 1 + \exp(\alpha + \beta \ln(q_x - 1)) \right]^{1/\nu} \quad \text{if } v < 0
   \]
   \[
   l_x = 1 - \left[ 1 - \exp(\alpha + \beta \ln(1 - q_x)) \right]^{1/\nu} \quad \text{if } v < 0
   \]
6. From the survivorship function, compute $45q_{15}$ and compare it with its desired level.

7. If they are not the same, change the input $\beta$ by a small amount and repeat Steps 4 to 6 until convergence is reached with respect to the level of $45q_{15}$.

This iterative method can easily be programmed on a spreadsheet. It may be noted that in Step 2, the estimated value of $v$ will always be negative. This is not a serious handicap because positive values would be required only when the mortality level is substantially different from the standard. Therefore, rather than changing $v$, the standard itself could be changed. But if a positive value for $v$ must be assumed, it can be done by skipping Step 2 and using the desired value of $v$ in Step 3 onwards. Further, it may also be noted that $v$ could also be fixed on the basis of $l_5$ of the standard life table. But fixing $v$ on the basis of the population’s $l_5$ is preferred because it causes a desired increase in $v$ as mortality levels fall.

One can use a similar procedure to fit the Gompertz model. In its case, $\alpha$ is estimated as

$$\alpha = \ln(-\ln(q_x)) - \beta \ln(-\ln(q_{xx}))$$

and the survivorship function is computed as

$$l_x = 1 - \exp(-\exp(\alpha + \beta \ln(-\ln(q_{xx}))))$$

To gauge the performance of this method, Table 2 presents results obtained in three cases, USA males 1900, China males, 1990 and India males 1986-90. For comparison, the results obtained using the WHO's modmatch methodology are also provided. In all cases, fitting was done using the estimates of $q_5$ and $45q_{15}$. The goodness of fit is checked using the root mean square error (RMSQE) in the logarithm of death rates, and the error is the estimates of $e_0$. To estimate $e_0$, the procedure of Coale and Guo (1989) for 'closing-out' the $l_x$ column has been followed. For maintaining comparability, this has been done in the case of both the reported as well as the fitted $l_x$ values.

Before commenting on the results obtained on individual cases, it must be noted that the errors in death rates and $e_0$ do not always indicate a similar degree of fit. In particular, the RMSQE of logarithm of death rates gives too large a weight to the fit at ages 10-20 where death rates are typically low. As the error in $e_0$ appears to be a better indicator of the overall fit, here it is relied upon in drawing conclusions.

In the case of USA 1900, the fitted survivorship function using the WHO methodology gave expectation of life at birth that was 0.7 years lower than the one derived from the actual survivorship function. In the applications of the Brass logit, generalized logit and Gompertz models three different standard $l_x$ functions were used. When the standard was either WHO global standard or USA of 1993, the error in $e_0$ was significantly higher than that obtained using the WHO methodology, though the indications from the percentage errors in the death rate were in some instances in the opposite direction. But in all cases, the generalized logit (with $v$ fixed at -0.47) and the Gompertz models provided fits that have smaller errors in $e_0$ than that given by the Brass logit model. But when the survivorship function of USA of 1920 was used as the standard, the fits obtained from the logit, generalized logit and Gompertz models were significantly better.
than that obtained through the WHO procedure, both in terms of life expectancy and age-specific death rates.

Similar results were obtained when the models were fitted to the mortality parameters of China for 1990 and India for 1986-90. In the case of China, when the WHO standard or that of South Korea of 1991 (representing the 'Far-Eastern' pattern) is used as the standard, errors in \( e_0 \) and death rates were higher than the errors for the fit obtained through the WHO procedure. But in every case, the generalized logit (\( v \) fixed at -0.22) and the Gompertz models performed better than the Brass logit model. When the survivorship function from the West model table (level 17) was used as the standard, the logit, generalized logit and Gompertz models implied errors in \( e_0 \) and death rates significantly lower than that obtained by fitting the data through the WHO mythology.

In the case of India of 1986-90, the logit-Gompertz models with the WHO life table or the West model tables as the standard performed worse than the WHO modmatch procedure. But when the South model table and especially the Indian life table of 1970-75 was used as the standard, the fit was significantly better than that provided by the WHO methodology.

Thus these applications suggest that the generalized logit and Gompertz models reduce the 'damage' resulting from the wrong choice of the standard \( l_x \) function. But even they could be providing poorer fits in comparison to the WHO methodology. But when the right choice for the standard has been made, the Brass logit model performs as good as other transformations, and provides better fit to the data than the WHO procedure. Thus it is recommended that all the three transformations be applied to a given case, and if they suggest significantly different fits (often indicated by \( \beta \) estimates of significantly different from 1), the standard should be changed.

**Preston's Integrated Procedure with Generalized Logit and Gompertz Models**

Another data circumstance often encountered is wherein an estimate of child mortality is available from a census or a survey, and also age distributions of the population from two enumerations. Preston (1983) has suggested a method of estimating a complete life table from these data that also produces an estimate of the birth rate. But his method is based on the assumption that a one-parameter logit transformation (i.e., \( \beta \) set to 1) is sufficient to describe mortality rates at ages above 5 years. This is a strong assumption that could produce a kink in the survivorship function at age 5. The models presented in this paper provide a better alternative.

It has been shown that the proportion of the population aged \( x \) in any closed population at time \( t \) can be written as

\[
c_x = b e^{-\int_a^x r_a \, da} l_x
\]

where \( r_a \) is the growth rate of the population in the age interval \( a \) to \( a+da \) during the time interval \( t \) to \( t+dt \) and \( b \) is the birth rate at time \( t \) (see Preston, Heuveline and Guillot 2001). For simplicity, the time subscript has been dropped from eq. (9).

From eq. (9) we have
From the generalized logit model defined in eq. (8) we have

\[ \ln[(1-l_x)^v - 1] = \alpha + \beta \ln[(1-l_x)^v - 1] \quad \text{for } v<0. \]

When \( v>0 \), the terms within the log function are reversed, as shown in eq. (8). Since one rarely needs to use this option, this case is not discussed further.

By substituting from eq. (10) we write,

\[
\ln \left( 1 - \frac{c_x e^{r_x a}}{b} \right)^v - 1 = \alpha + \beta \ln[(1-l_x)^v - 1]. \tag{11}
\]

In eq. (11) there are four unknowns, \( b, v, \alpha \) and \( \beta \). If \( b \) and \( v \) are known, it is possible to estimate \( \alpha \) and \( \beta \) from this equation by fitting a straight line to the term in the left-hand side of the equation and the term in the right involving the standard \( l_x \) function. Because there is an independent information on \( q_5 \), we also have a restriction that

\[ \ln(q_5^v - 1) = \alpha + \beta \ln(q_{s5}^v - 1) \]

So, if \( v \) can be appropriately fixed, \( b \) can be estimated. We could proceed as follows:

1. Choose an appropriate standard life table.
2. Fix \( v \) from the independent estimate of \( q_5 \) using the relation
   \[ v = \frac{\ln 2}{\ln(q_5)}. \]
3. Assume a staring value for \( b \).
4. Fit the line defined by eq. (11) and estimate \( \alpha \) and \( \beta \).
5. Estimate \( q_5 \) using the relation
   \[ q_5 = \left[ 1 + \exp(\alpha + \beta \ln(q_{s5}^v - 1)) \right]^\frac{1}{v} \quad \text{for } v<0. \]
6. Compare the resulting \( q_5 \) with the independently derived estimate of \( q_5 \).
7. If they are not the same, change the input \( b \) by a small amount and repeat Steps 4 to 6 until convergence is reached with respect to the level of \( q_5 \).

It is to be noted that convergence would be reached even if \( v \) is set without using the estimate of \( q_5 \). If it is fixed at -1, the model would be the Brass logit system.
A similar method can be followed in the case of the Gompertz model. From eq. (4) and eq. (10) we have:

\[
\ln\left(-\ln\left(1 - \frac{c_2e^{\int_{r_a}^{r_b} dr_a}}{b}\right)\right) = \alpha + \beta \ln(-\ln(q_s))
\]  

(12)

The recommended fitting procedure is essentially the same as above, except that Step 2 is skipped (since \(v\) is implicitly fixed at 0), eq. (12) is used for fitting the line instead of eq. (11), and \(q_5\) is estimated using the relation

\[
q_5 = \exp(-\exp(\alpha + \beta \ln(-\ln(q_s)))).
\]

As an illustration, Table 3 shows the details of the computation for Indian females 1961-71, an example also used by Preston (1983). It shows the results of the final iteration using a birth rate of 0.0417 that produces the independent estimate of \(q_5\) of 0.224. Figure 5 shows the plot of the data points used for fitting the line. As can be seen, unlike the original method, the data points do not muddle in one end of the line (see Preston, 1983:219). To fit the line, the group-mean procedure has been used with data for age 5, 10, and 15 forming one group and age 20 to 70 forming the other group.

Table 4 gives the summary of results obtained using different models and standard life tables. It is interesting to note that when the same standard is used, logit (with \(v\) set at -1), generalized logit and Gompertz models produce very similar estimates of birth rate, adult mortality (\(45q_{15}\)) and life expectancy, though the errors around the line are minimized more by the Gompertz and the generalized logit versions. But the estimates show greater sensitivity to the choice of the standard survivorship function. Interestingly, model fits are better, and estimates of \(\beta\) are closer to 1, with the West model life table (level 13) than either with the South model table or with the Indian life table of 1970-75. This supports the possibility of a switch in the age pattern of mortality in India that coincided with the declines in malarial mortality and deaths due to starvation (see Bhat 1989). When the West model life table is used, the estimate of \(e_5\) is around 52 years, \(45q_{15}\) is 0.43 and birth rate is 41.7 per 1000. Preston's estimates, obtained using the South model life table as the standard, were \(e_5\) of 51 years and birth rate of 42.1. These estimates are closer to what I also get using the South model life table (Table 4). Although differences are not that large, these estimates are improbable because they imply \(\beta\) in the range of 1.4-1.5, which remained disguised in Preston's procedure. It should also be noted that though the estimates of \(e_5\) obtained using the South model table is about one year lower than that given by the West model table, the estimate of \(45q_{15}\) from the latter is marginally higher than that from the former.

A comment is also in order here on the relative performance of the generalized logit model and the Gompertz model. In several instances, the Gompertz model was shown to be performing better than the generalized logit. But this is simply because the method suggested for fixing \(v\) implied values lower than zero. If \(v\) were to be fixed nearer to zero, or even at positive values, the difference in performance would have disappeared, or even reversed. The generalized
logit is the preferred model because it is more flexible than the Gompertz model, though information may not be available in sufficient detail to estimate the parameter $v$. With regard to the WHO methodology, it is found to perform better when the mortality level of the population is far removed from the standard. As a general approach to modelling age patterns of mortality, it performance could improve if the standard pattern of deviations were recomputed using either the Gompertz model, or the generalized logit model with appropriately fixed $v$, as the base.

References


Figure 1. Density Distribution of Deaths in Female West Model Life Tables When the Age Scale is Transformed to the Logits of Standard Survivorship Function

**Standard lx: e0 = 45**

- Deaths (dx) for a unit change in standardized age scale

**Standard lx: e0 = 75**

- Deaths (dx) for a unit change in standardized age scale
Figure 2. Density Distribution of Deaths in US Male Life Tables of Selected Years When the Age Scale is Transformed to the Logits of Standard Survivorship Function

### Standard lx: 1900

Deaths (dx) for a unit change in the standardized age scale

### Standard lx: 1993

Deaths (dx) for a unit change in the standardized age scale
Figure 3. Effect of Changing the Model Parameters in the Generalized Logit Model

Effect of $v$ when $b$ is changed

Effect of change in $b$ when $a$ and $v$ are constant (with $v$ chosen such that curves intersect at age 5)

Effect of $v$ when $a$ is changed
Figure 4. Transformed Survivorship Functions of US Males, 1910-1993 Plotted Against Similarly Transformed Survivorship Functions of US Males, 1900 Under Generalized Logit and Gompertz Models

$v = -1$

$v = -0.01$
$\nu = 0.5$

Transformed $q_x$, 1900

Transformed $q_x$ of years after 1900

$\log(-\log(q_x))$

Transformed $q_x$, 1900

Transformed $q_x$ of years after 1900
Figure 5. Plot of the Data Points Used for Fitting the Line Under the Integrated Procedure with Generalized Logit Model, Indian Females, 1961-71.

Table 1: Results of Fitting the Survivorship Function of USA 1993 Males Under Different Models Through Linear Regression

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Parameter estimates</th>
<th>N</th>
<th>$R^2$</th>
<th>$F$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$B$</td>
<td></td>
<td></td>
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<tr>
<td>Standard $l$: USA 1900</td>
<td>Logit, $q_x$</td>
<td>2.28</td>
<td>1.41</td>
<td>18</td>
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<td>Gompertz, $q_x$</td>
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<td>0.85</td>
<td>18</td>
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<td>Generalized logit, $v = 0.5$</td>
<td>0.32</td>
<td>0.64</td>
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<tr>
<td>Standard $l$: USA 1960</td>
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<td>1.07</td>
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<td>Gompertz, $q_x$</td>
<td>0.34</td>
<td>0.87</td>
<td>18</td>
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<tr>
<td></td>
<td>Generalized logit, $v = 0.5$</td>
<td>0.01</td>
<td>0.77</td>
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</table>
Table 2: Results of Fitting Models with Estimates of Child and Adult Mortality, USA 1900, China, 1990 and India 1986-90.

<table>
<thead>
<tr>
<th>Country, Year</th>
<th>Standard lx</th>
<th>Model Parameter estimates</th>
<th>RMSQE</th>
<th>Error in</th>
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<tr>
<td>USA, 1900, Males</td>
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<td>WHO modmatch</td>
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<td>USA, 1993</td>
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<td>-1.29</td>
<td>0.65</td>
<td>-0.47</td>
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<td>-1.04</td>
<td>0.96</td>
<td>(0.0)</td>
</tr>
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<td>USA, 1920</td>
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<td>0.89</td>
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<td>-0.43</td>
<td>0.95</td>
<td>-0.47</td>
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<td>0.64</td>
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<tr>
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<td>Gompertz qx</td>
<td>0.30</td>
<td>0.93</td>
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<td>WHO modmatch</td>
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<tr>
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<td>1.13</td>
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<tr>
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<tr>
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<td>Gompertz qx</td>
<td>0.27</td>
<td>0.93</td>
<td>(0.0)</td>
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<table>
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<tr>
<th>Age interval</th>
<th>Intercensal cumulated growth rates</th>
<th>Cumulated growth rates</th>
<th>Standard $l_x$ (West, level 13)</th>
<th>Generalized logit model Y-values</th>
<th>Gompertz model Y-values</th>
</tr>
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<tbody>
<tr>
<td>$x - x+5$</td>
<td>$lx$</td>
<td>$cx$</td>
<td>$bl_x$</td>
<td>$y-values$</td>
<td>$x-values$</td>
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<td>1.0000</td>
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<tr>
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<td>0.0900</td>
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<td>0.8185</td>
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<td>0.2060</td>
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<td>1.8125</td>
<td>0.0008</td>
<td>0.1442</td>
<td>-2.5925</td>
</tr>
</tbody>
</table>

\[ bl_x = c_x \exp \left( \sum_{y=0}^{x-5} r_y \right). \]

" with $b = 0.0417.$


<table>
<thead>
<tr>
<th>Standard $l_x$</th>
<th>Model parameters</th>
<th>Computed statistics</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\alpha$ $\beta$ $v$</td>
<td>$b$ $e_0$ $e_5$ $45q_{15}$ RMSQE*</td>
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<tr>
<td>West model, Logit $q_x$ level 13</td>
<td>-0.36 1.07 (-1)</td>
<td>41.7 44.4 51.9 0.435 0.111</td>
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<tr>
<td>South model, Logit $q_x$ level 13</td>
<td>-0.55 1.46 (-1)</td>
<td>41.8 43.7 51.1 0.429 0.121</td>
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<tr>
<td>India, Logit $q_x$ 1970-75</td>
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<td>41.9 43.5 50.7 0.446 0.172</td>
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<td>41.9 43.8 51.1 0.434 0.110</td>
</tr>
</tbody>
</table>

* For the fitted line.